9. Integracija nekih transcedentnih (nealgebarskih) f-ja

U ovoj lekciji ćemo posmatrati sljedeće tipove integrala (R predstavlja racionalnu f-ju)

I. $\int R(\sin x, \cos x)\,dx$ – uvodimo smjenu $z = tg\frac{x}{2}$,

pri čemu je $\sin x = \dfrac{2z}{1+z^2}$, $\cos x = \dfrac{1-z^2}{1+z^2}$, $dx = \dfrac{2dz}{1+z^2}$

II. $\int R(tg\,x)\,dx$ – uvodimo smjenu $tg\,x = z$,

pri čemu je $x = arc\,tg\,z$, $dx = \dfrac{dz}{1+z^2}$

III. $\int R(e^x)\,dx$ – uvodimo smjenu $e^x = z$,

pri čemu je $x = \ln z$, $dx = \dfrac{dz}{z}$.

$$z = tg\frac{x}{2} \Rightarrow \frac{x}{2} = arc\,tg\,z \Rightarrow x = 2\,arc\,tg\,z$$

$$dx = \frac{2\,dz}{1+z^2}, \quad \sin x = \sin 2\cdot\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \begin{array}{l} /:\cos^2\frac{x}{2} \\ /:\cos^2\frac{x}{2}\end{array} =$$

$$= \frac{2\,tg\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{2z}{1+z^2} \quad tj. \quad \sin x = \frac{2z}{1+z^2}$$

$$\cos x = \cos 2\cdot\frac{x}{2} = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \begin{array}{l} /:\cos^2\frac{x}{2} \\ /:\cos^2\frac{x}{2}\end{array} = \frac{1-z^2}{1+z^2}$$

Odrediti integrale

a) $\displaystyle\int \frac{dx}{2\sin x - \cos x}$; b) $\displaystyle\int \frac{dx}{5+4\cos ax}$;

c) $\displaystyle\int \frac{\operatorname{tg} x\, dx}{1-\operatorname{ctg}^2 x}$; d) $\displaystyle\int \frac{e^{3x}\, dx}{e^{2x}+1}$.

Rj.

a)
$$\int \frac{dx}{2\sin x - \cos x} = \left| \begin{array}{ll} \operatorname{tg}\frac{x}{2} = z & \sin x = \frac{2z}{1+z^2} \\ dx = \frac{2\,dz}{1+z^2} & \cos x = \frac{1-z^2}{1+z^2} \end{array} \right| =$$

$$= \int \frac{\frac{2\,dz}{1+z^2}}{\frac{4z}{1+z^2} - \frac{1-z^2}{1+z^2}} = \int \frac{2\,dz}{z^2+4z-1} \overset{(*)}{=} 2\int \frac{d(z+2)}{(z+2)^2-5} =$$

$$z^2+4z-1 = z^2+2\cdot z\cdot 2+4-4-1 = (z+2)^2-5$$
$$d(z+2) = dz \qquad \ldots(*)$$

$$= 2\cdot\frac{1}{2\sqrt5}\ln\left|\frac{z+2-\sqrt5}{z+2+\sqrt5}\right| + C = \frac{1}{\sqrt5}\ln\left|\frac{\operatorname{tg}\frac{x}{2}+2-\sqrt5}{\operatorname{tg}\frac{x}{2}+2+\sqrt5}\right| + C$$

b)
$$\int \frac{dx}{5+4\cos ax} = \left| \begin{array}{ll} \operatorname{tg}\frac{ax}{2} = z \Rightarrow \frac{ax}{2} = \operatorname{arctg} z & \\ dx = \frac{2\,dz}{a(1+z^2)}, & \cos ax = \frac{1-z^2}{1+z^2} \end{array} \right| =$$

$$= \int \frac{\frac{2\,dz}{a(1+z^2)}}{5+\frac{4(1-z^2)}{1+z^2}} = \frac{2}{a}\int \frac{dz}{5+5z^2+4-4z^2} = \frac{2}{a}\int \frac{dz}{z^2+9} =$$

$$= \frac{2}{a} \cdot \frac{1}{3} \operatorname{arctg} \frac{z}{3} + c = \frac{2}{3a} \operatorname{ar} \operatorname{tg}\left(\frac{1}{3} \operatorname{tg} \frac{ax}{2}\right) + c$$

c) $$\int \frac{\operatorname{tg} x \, dx}{1 - \operatorname{ctg}^2 x} = \left| \begin{array}{ll} \operatorname{tg} x = z, & x = \operatorname{arctg} z \\ dx = \frac{dz}{1+z^2} & \operatorname{ctg}^3 x = \left(\frac{1}{\operatorname{tg} x}\right)^2 = \frac{1}{z^2} \end{array} \right|$$

$$= \int \frac{\frac{z \, dz}{1+z^2}}{1 - \frac{1}{z^2}} = \int \frac{\frac{z \, dz}{1+z^2}}{\frac{z^2-1}{z^2}} = \int \frac{z^3 \, dz}{z^4 - 1} =$$

$$= \int \frac{\frac{1}{4} d(z^4 - 1)}{z^4 - 1} = \frac{1}{4} \ln |z^4 - 1| + c = \frac{1}{4} \ln |\operatorname{tg}^4 x - 1| + c$$

d) $$\int \frac{e^{3x} \, dx}{e^{2x} + 1} = \left| \begin{array}{l} e^x = z \\ e^x dx = dz \\ dx = \frac{dz}{z} \end{array} \right| = \int \frac{z^3 \cdot \frac{dz}{z}}{z^2 + 1} =$$

$$= \int \frac{z^2 \overset{+1-1}{} dz}{z^2 + 1} = \int \left(1 - \frac{1}{z^2+1}\right) dz = \int dz - \int \frac{dz}{z^2+1}$$

$$= z - \operatorname{arctg} z + c = e^x - \operatorname{arctg} e^x + c$$

Zadaci za vježbu

1. $\displaystyle\int \frac{\cos x \, dx}{1+\cos x}$

2. $\displaystyle\int \frac{dx}{\sin kx}$

3. $\displaystyle\int \frac{dx}{\sin^3 x}$

4. $\displaystyle\int \frac{dx}{4\cos x + 3\sin x}$

5. $\displaystyle\int tg^5 3x \, dx$

6. $\displaystyle\int \frac{dx}{1+tg\, x}$

7. $\displaystyle\int \frac{e^{2t}-2e^t}{1+e^{2t}}\, dt$

8. $\displaystyle\int \frac{e^x-1}{e^x+1}\, dx$

9. * $\displaystyle\int \frac{1+tg\, x}{\sin 2x}\, dx$

10. * $\displaystyle\int \frac{e^{2x}\, dx}{(2+e^x+e^{-x})^2}$

Rješenja

1. $x - tg\frac{x}{2}$

2. $\frac{1}{k}\left| tg \frac{kx}{2}\right|$

3. $\frac{1}{2}\left(\ln\left|tg\frac{x}{2}\right| - \frac{\cos x}{\sin^2 x}\right)$

4. $\frac{1}{5}\ln\left|\frac{1+2\,tg\frac{x}{2}}{2-tg\frac{x}{2}}\right|$

5. $\frac{1}{12}tg^4 3x - \frac{1}{6}tg^2 3x - \frac{1}{3}\ln|\cos 3x|$

6. $\frac{1}{2}(x+\ln|\sin x+\cos x|)$

7. $\frac{1}{2}\ln(e^{2t}+1) - 2\,\mathrm{arc\,tg}\, e^t$

8. $2\ln(e^x+1)-x$

9. $\frac{1}{2}(tg\, x+\ln|tg\, x|)$

10. $\ln(e^x+1)+\frac{18e^{2x}+27e^x+11}{6(e^x+1)^3}$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija nekih nealgebarskih funkcija)

$\int R(\sin x, \cos x)\,dx$, R - racionalna f-ja

koristimo smjenu $tg\frac{x}{2} = t \Rightarrow$

$\Rightarrow dx = \frac{2\,dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$tg\frac{x}{2} = t$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2} \quad :\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2} \quad :\cos^2\frac{x}{2}} = \frac{2\,tg\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2} \quad :\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2} \quad :\cos^2\frac{x}{2}} = \frac{1-tg^2\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{1-t^2}{1+t^2}$$

$$tg\frac{x}{2} = t \Rightarrow \frac{x}{2} = arctg\,t \Rightarrow x = 2arctg\,t \Rightarrow dx = \frac{2\,dt}{1+t^2}$$

1.
$$\int\frac{dx}{\sin x} = \left|\begin{array}{l} tg\frac{x}{2} = t \\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2}\end{array}\right| = \int\frac{\frac{2\,dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int\frac{dt}{t} = \ln|t| + C =$$

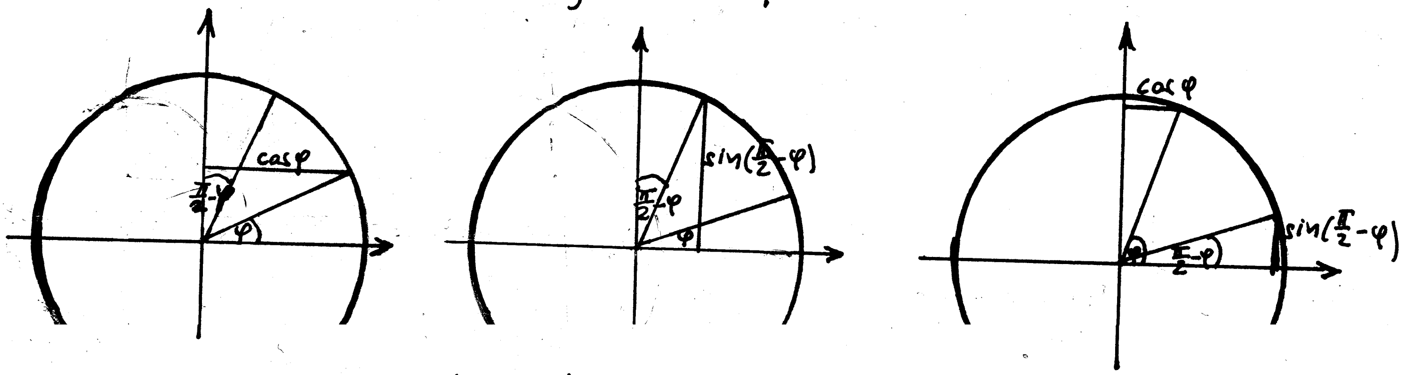
$$= \ln\left|tg\frac{x}{2}\right| + C$$

2.
$$\int\frac{dx}{\cos x} = \left|\begin{array}{c} \sin x = \cos(\frac{\pi}{2}-x) \\ \cos x = \sin(\frac{\pi}{2}-x) \\ \text{OBJASNI} \\ \text{OVO}\end{array}\right| = \int\frac{dx}{\sin(\frac{\pi}{2}-x)} = \left|\begin{array}{l} \frac{\pi}{2}-x = t \\ -dx = dt \\ dx = -dt\end{array}\right| =$$

$$= -\int\frac{dt}{\sin t} \overset{1.zad.}{=\!=} -\ln\left|tg\frac{t}{2}\right| + C = -\ln\left|tg\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C =$$

$$= \ln\left|tg\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|^{-1} = \ln\left|ctg\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C = \left| \begin{aligned} & ctg\,x = \frac{\cos x}{\sin x} = \\ & = \frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)} = tg\left(\frac{\pi}{2}-x\right) \end{aligned} \right|$$

$$= \ln\left|tg\,\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C = \ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$$



3.

$$\int \frac{dx}{5-4\sin x+3\cos x} = \left| \begin{aligned} & tg\frac{x}{2} = t \\ & dx = \frac{2dt}{1+t^2} \\ & \sin x = \frac{2t}{1+t^2} \\ & \cos x = \frac{1-t^2}{1+t^2} \end{aligned} \right| = \int \frac{\frac{2dt}{1+t^2}}{5-4\cdot\frac{2t}{1+t^2}+3\cdot\frac{1-t^2}{1+t^2}} =$$

$$= 2\int \frac{\cancel{\frac{dt}{1+t^2}}}{\frac{5+5t^2-8t+3-3t^2}{\cancel{1+t^2}}} = 2\int \frac{dt}{2t^2-8t+8} = \int \frac{dt}{t^2-4t+4}$$

$$= \int \frac{dt}{(t-2)^2} = \left| \begin{aligned} & t-2 = z \\ & dt = dz \end{aligned} \right| = \int \frac{dz}{z^2} = -\frac{1}{z} + C = -\frac{1}{t-2} + C = -\frac{1}{tg\frac{x}{2}-2} + C$$

4.

$$I = \int \frac{\cos x + 2\sin x}{4\cos x + 3\sin x}\,dx = \left| \begin{aligned} & tg\frac{x}{2} = t \qquad && \cos x = \frac{1-t^2}{1+t^2} \\ & dx = \frac{2dt}{1+t^2} && \sin x = \frac{2t}{1+t^2} \end{aligned} \right| =$$

$$= \int \frac{\frac{1-t^2}{1+t^2}+2\,\frac{2t}{1+t^2}}{4\cdot\frac{1-t^2}{1+t^2}+3\,\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2} = \int \frac{\frac{1-t^2+4t}{1+t^2}}{\frac{4-4t^2+6t}{1+t^2}}\cdot\frac{2dt}{1+t^2} =$$

$$= \int \frac{-t^2+4t+1}{(-2t^2+3t+2)(1+t^2)}\,dt = \ldots$$

5. $$\int \frac{dx}{8-4\sin x+7\cos x}$$

6. $$\int \frac{\cos x+\sin x}{\cos x-2\sin x}\,dx$$

$$\int R(\sin^2 x, \sin x \cos x, \cos^2 x)\, dx$$ R – racionalna f-ja

ili $\int R(tg\, x)\, dx$

uvodimo smjenu $tg\, x = t \Rightarrow$

$\Rightarrow dx = \frac{dt}{1+t^2}$, $\sin^2 x = \frac{t^2}{1+t^2}$, $\cos^2 x = \frac{1}{1+t^2}$,

$$\sin x \cos x = \frac{t}{1+t^2}$$

$tg\, x = t \Rightarrow x = arc\, tg\, t \Rightarrow dx = \frac{dt}{1+t^2}$

$$\sin^2 x = \frac{\sin^2 x \quad :\cos^2 x}{\sin^2 x + \cos^2 x \quad :\cos^2 x} = \frac{tg^2 x}{tg^2 x + 1} = \frac{t^2}{1+t^2}$$

$$\cos^2 x = \frac{\cos^2 x \quad :\cos^2 x}{\sin^2 x + \cos^2 x \quad :\cos^2 x} = \frac{1}{tg^2 x + 1} = \frac{1}{1+t^2}$$

$$\sin x \cos x = \frac{\sin x \cos x \quad :\cos^2 x}{\sin^2 x + \cos^2 x \quad :\cos^2 x} = \frac{tg\, x}{tg^2 x + 1} = \frac{t}{1+t^2}$$

1. $\int \frac{dx}{\cos^4 x} = \left| \begin{matrix} tg\, x = t & \cos^2 x = \frac{1}{1+t^2} \\ dx = \frac{dt}{1+t^2} & \end{matrix} \right| = \int \frac{\frac{dt}{1+t^2}}{\left(\frac{1}{1+t^2}\right)^2} = \int \frac{(1+t^2)^{\cancel{2}}}{\cancel{1+t^2}} dt =$

$$= \int (1+t^2)\, dt = \int dt + \int t^2\, dt = t + \frac{t^3}{3} + C = tg\, x + \frac{1}{3} tg^3 x + C$$

2. $$\int \frac{dx}{\sin^2 x - 4\sin x\cos x + 5\cos^2 x} = \left| \begin{matrix} tg\,x = t & dx = \frac{dt}{1+t^2} & \sin x\cdot\cos x = \\ \sin^2 x = \frac{t^2}{1+t^2} & \cos^2 x = \frac{1}{1+t^2} & = \frac{t}{1+t^2} \end{matrix} \right|$$

$$= \int \frac{\frac{dt}{1+t^2}}{\frac{t^2}{1+t^2} - 4\cdot\frac{t}{1+t^2} + 5\cdot\frac{1}{1+t^2}} = \int \frac{dt}{t^2-4t+5} = \int \frac{dt}{(t-2)^2+1} = \operatorname{arctg}(t-2) + C$$

$$= \operatorname{arc\,tg}(tg\,x - 2) + C$$

3. $$\int tg^3 x\, dx = \left| \begin{matrix} tg\,x = t \\ dx = \frac{dt}{1+t^2} \end{matrix} \right| = \int t^3 \cdot \frac{dt}{1+t^2} = \int \frac{t^3 + t - t}{1+t^2}\, dt = \int \frac{t+t^3}{1+t^2} dt -$$

$$- \int \frac{t}{1+t^2} dt = \int \frac{\cancel{t(1+t^2)}}{\cancel{(1+t^2)}} dt - \frac{1}{2}\int \frac{2t\,dt}{1+t^2} = \left| \begin{matrix} t^2 = s \\ 2t\,dt = ds \end{matrix} \right| =$$

$$= \int t\,dt - \frac{1}{2}\int \frac{ds}{1+s} = \frac{t^2}{2} - \frac{1}{2}\ln|1+s| + C = \frac{1}{2}t^2 - \frac{1}{2}\ln|t^2+1| + C =$$

$$= \frac{1}{2}tg^2 x - \frac{1}{2}\ln|tg^2+1| + C = \frac{1}{2}tg^2 x - \frac{1}{2}\ln\left|\frac{\sin^2 x}{\cos^2 x} + 1\right| + C = \frac{1}{2}tg^2 x - \frac{1}{2}\ln\left|\frac{1}{\cos^2 x}\right| + C$$

$$= \frac{1}{2}tg^2 x + \ln\left|\frac{1}{\cos^2 x}\right|^{-\frac{1}{2}} + C = \frac{1}{2}tg^2 x + \ln|\cos^2 x|^{\frac{1}{2}} + C = \frac{1}{2}tg^2 x + \ln|\cos x| + C$$

4. ✓ $$\int \frac{\cos x + 2\sin x \quad :\cos x}{4\cos x + 3\sin x \quad :\cos x}\, dx = \int \frac{1+2\,tg\,x}{4+3\,tg\,x} dx = \left| \begin{matrix} tg\,x = t \\ dx = \frac{dt}{1+t^2} \end{matrix} \right| = \int \frac{1+2t}{4+3t}\cdot\frac{dt}{1+t^2}$$

$$\frac{1+2t}{(4+3t)(1+t^2)} = \frac{a}{4+3t} + \frac{bt+c}{1+t^2}$$

...

5. ✓ $$\int \frac{dx}{\sin^4 x}$$

6. ✓ $$\int \frac{dx}{3\cos^2 x + 4\sin^2 x}$$

7. ✓ $$\int \frac{tg\,x}{tg^2 x - 2\,tg\,x - 3}\, dx$$ → Rj. $-\frac{1}{10}x + \frac{3}{40}\ln|tg\,x - 3| + \frac{1}{8}\ln|tg\,x + 1| + \frac{1}{5}\ln|\cos x| + C$

(#) Izračunati integral $I=\int\frac{dx}{3\cos^2x+4\sin^2x}$.

Rj. $tg\,x=t$

$x=arc\,tg\,t$

$dx=\frac{dt}{1+t^2}$

$$\sin^2x=\frac{\sin^2x}{\sin^2x+\cos^2x}\,\begin{matrix}:\cos^2x\\:\cos^2x\end{matrix}=\frac{tg^2x}{tg^2x+1}=\frac{t^2}{1+t^2}$$

$$\cos^2x=\frac{\cos^2x}{\sin^2x+\cos^2x}\,\begin{matrix}:\cos^2x\\:\cos^2x\end{matrix}=\frac{1}{tg^2x+1}=\frac{1}{1+t^2}$$

$$I=\int\frac{dx}{3\cos^2x+4\sin^2x}=\left|\begin{array}{ll} tg\,x=t & \sin^2x=\frac{t^2}{1+t^2}\\ dx=\frac{dt}{1+t^2} & \cos^2x=\frac{1}{1+t^2}\end{array}\right|=$$

$$=\int\frac{\frac{dt}{1+t^2}}{\frac{3}{1+t^2}+\frac{4t^2}{1+t^2}}=\int\frac{\frac{dt}{1+t^2}}{\frac{3+4t^2}{1+t^2}}=\int\frac{dt}{3+4t^2}=\int\frac{dt}{(\sqrt3)^2+(2t)^2}$$

$$=\left|\begin{array}{l}2t=\sqrt3\,u\\ 2dt=\sqrt3\,du\\ dt=\frac{\sqrt3}{2}du\\ u=\frac{2t}{\sqrt3}\end{array}\right|=\int\frac{\frac{\sqrt3}{2}du}{3+3u^2}=\frac{\sqrt3}{2}\cdot\frac13\int\frac{du}{1+u^2}=\frac{\sqrt3}{6}\,arc\,tg\,u+C=$$

$$=\frac{\sqrt3}{6}\,arc\,tg\,\frac{2t}{\sqrt3}+C=\frac{\sqrt3}{6}\,arc\,tg\,\frac{2\,tg\,x}{\sqrt3}+C$$

Odrediti $I=\int x^2\sin x\,dx$.

Rj.

$$I=\int x^2\sin x\,dx=\begin{vmatrix} u=x^2 & dv=\sin x\,dx \\ du=2x\,dx & v=-\cos x \end{vmatrix}=$$

$$=-x^2\cos x-\int(-\cos x)\cdot 2x\,dx=-x^2+2\int x\cos x\,dx$$

$$\int x\cos x\,dx=\begin{vmatrix} u=x & dv=\cos x\,dx \\ du=dx & v=\sin x \end{vmatrix}=x\sin x-\int\sin x\,dx$$

$$=x\sin x-(-\cos x)+C=x\sin x+\cos x+C_1$$

$$I=-x^2\cos x+2(x\sin x+\cos x+C_1)=-x^2\cos x+2x\sin x+2\cos x+C$$

$$=2x\sin x-(x^2-2)\cos x+C$$

Izračunati integral $\int\frac{1-\sin x+\cos x}{1+\sin x-\cos x}\,dx$.

Rj.

Uvodimo smjenu $\operatorname{tg}\frac{x}{2}=t \Rightarrow \frac{x}{2}=\operatorname{arc\,tg} t$

$$x=2\operatorname{arc\,tg} t$$

$$dx=\frac{2}{1+t^2}$$

$\sin 2x=2\sin x\cos x$

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=$$

$$=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\ \begin{matrix}/:\cos^2\frac{x}{2}\\ /:\cos^2\frac{x}{2}\end{matrix}=\frac{2\operatorname{tg}\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1}=\frac{2t}{t^2+1}$$

$\cos 2x=\cos^2 x-\sin^2 x$

$$\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\ \begin{matrix}/:\cos^2\frac{x}{2}\\ /:\cos^2\frac{x}{2}\end{matrix}=$$

$$=\frac{1-\operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1}=\frac{1-t^2}{1+t^2}$$

$$\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\,dx = \left| \begin{matrix} tg\frac{x}{2}=t & \sin x = \frac{2t}{1+t^2} \\ dx=\frac{2}{1+t^2} & \cos x = \frac{1-t^2}{1+t^2} \\ x = 2\,arc\,tg\,t & \end{matrix} \right| =$$

$$= \int \frac{1-\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{\frac{\cancel{1}+\cancel{t^2}-2t+1-\cancel{t^2}}{\cancel{1+t^2}}}{\frac{\cancel{1}+t^2+2t-\cancel{1}+t^2}{\cancel{1+t^2}}} \cdot \frac{2}{1+t^2}\,dt$$

$$= 2\int \frac{2-2t}{2t^2+2t}\cdot\frac{1}{1+t^2}\,dt = 2\int \frac{1-t}{(t^2+t)(1+t^2)}\,dt = 2\int \frac{1-t}{t(t+1)(t^2+1)}\,dt$$

$$\frac{1-t}{t(t+1)(t^2+1)} = \frac{A}{t}+\frac{B}{t+1}+\frac{Ct+D}{t^2+1} \quad \Big/ \cdot t(t+1)(t^2+1)$$

$$-t+1 = A\underbrace{(t+1)(t^2+1)}_{t^3+t^2+t+1} + B\underbrace{(t^2+1)\cdot t}_{t^3+t} + (Ct+D)\underbrace{t(t+1)}_{t^2+t}$$

$$-t+1 = A(t^3+t^2+t+1)+B(t^3+t)+C(t^3+t^2)+D(t^2+t)$$

$$\begin{aligned} A+B+C \quad &= 0 \\ A+C+D &= 0 \\ A+B \quad +D &= -1 \\ A \quad &= 1 \end{aligned}$$

$$\begin{aligned} B+C \quad &= -1 \quad (a) \\ C+D &= -1 \quad (b) \\ B \quad +D &= -2 \quad (c) \end{aligned}$$

$$(a):\ B+C=-1$$
$$(c)-(b):\ B-C=-1 \quad +$$
$$2B=-2$$
$$B=-1$$

$$-1+D=-2$$
$$D=-1$$

$$C-1=-1$$
$$C=0$$

$$A=1 \qquad C=0$$
$$B=-1 \qquad D=-1$$

$$2\int \frac{1-t}{t(t+1)(t^2+1)}\,dt = 2\int\left(\frac{1}{t}+\frac{(-1)}{t+1}+\frac{(-1)}{t^2+1}\right)dt =$$

$$= 2\ln|t| - 2\ln|t+1| - 2\,arc\,tg\,t + C =$$

$$= 2\ln\left|tg\frac{x}{2}\right| - 2\ln\left|tg\frac{x}{2}+1\right| - 2\,arc\,tg\left|tg\frac{x}{2}\right| + C$$

Dio tablice integrala

1. $\int u^a\,du = \frac{u^{a+1}}{a+1} + C,\ \ a \neq -1.$

2. $\int u^{-1}\,du = \int \frac{du}{u} = \int \frac{u'}{u}\,dx =$
$= \ln|u| + C.$

3. $\int a^u\,du = \frac{a^u}{\ln a} + C;\ \int e^u\,du =$
$= e^u + C.$

4. $\int \sin u\,du = -\cos u + C.$

5. $\int \cos u\,du = \sin u + C.$

6. $\int \sec^2 u\,du = \operatorname{tg} u + C.$

7. $\int \operatorname{cosec}^2 u\,du = -\operatorname{ctg} u + C.$

8. $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \operatorname{arc\,tg} \frac{u}{a} + C.$

9. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C.$

10. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C.$

11. $\int \frac{du}{\sqrt{u^2 + a}} =$
$= \ln \left| u + \sqrt{u^2 + a} \right| + C.$